Problem Sheet 9

Throughout this problem sheet let b, σ satisfy the Lipschitz and linear growth constraints for the existence and uniqueness of the associated SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{1}$$

with $X_0 \in L^2(\Omega; \mathbb{R})$ independent of W.

Exercise 9.1

Let X be the unique solution of (1). Prove that for any $T \geq 0$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t|^2\right) < \infty.$$

Exercise 9.2

Denote by X^ε the solution of

$$dX_t^\varepsilon = b(t, X_t^\varepsilon)dt + \varepsilon\sigma(t, X_t^\varepsilon)dW_t$$

and X the solution of

$$dX_t = b(t, X_t)dt$$

with the same initial condition X_0. Show that for any $T \geq 0$,

$$\lim_{\varepsilon\to 0} \mathbb{E}\left(\sup_{t\in[0,T]} |X_t^\varepsilon - X_t|^2\right) = 0.$$

Exercise 9.3

Let $X_0 \in L^\infty(\Omega; \mathbb{R})$ and X the unique solution of (1). Prove that for a given $0 < \alpha < \frac{1}{2}$, there exists a modification $\tilde{X}$ of X such that for $\mathbb{P} - a.e.\ \omega$, $\tilde{X}_\cdot(\omega) \in C^\alpha$.

Exercise 9.4

Let $f \in L^2([0,t]; \mathbb{R})$, $\lambda \geq 0$, and define

$$\xi = \int_0^t f_s dW_s, \qquad Z_s = e^{\lambda \int_0^s f_r dW_r - \frac{\lambda^2}{2}\int_0^s f_r^2 dr}.$$

1. **Bonus Question:** Prove that Z is a genuine martingale on $[0,t]$.
2. Deduce that $\mathbb{E}\left(e^{\lambda\xi}\right) = e^{\frac{\lambda^2}{2}\int_0^t f_r^2 dr}$.
3. Further deduce that $\xi \sim N\left(0, \int_0^t f_s^2 ds\right)$.
4. If we replace f with a random progressively measurable $L^2(\Omega \times [0,t]; \mathbb{R})$ process, is ξ still normally distributed?

Exercise 9.5

Consider the simplified SDE

$$dX_t = b(X_t)dt + dW_t$$

with unique solution X, and the Euler-Maruyama Approximation as follows. Fix a time interval $[0,T]$ and $N > 0$, defining $h := \frac{T}{N} < 1$ and the times $t_n := hn$ for $0 \le n \le N$. Furthermore recursively define

$$X_{n+1} = X_n + hb(X_n) + W_{t_{n+1}} - W_{t_n}$$

with X_0 as given.

1. Show that there exists a constant C such that for all $t \in [t_n, t_{n+1})$,

$$\mathbb{E}\left(|X_t - X_{t_n}|^2\right) \le Ch.$$

2. Prove that there exists a constant C such that for all n,

$$\mathbb{E}\left(|X_{t_n} - X_n|^2\right) \le (1+h)\left[(1+Ch)\,\mathbb{E}\left(|X_{t_{n-1}} - X_{n-1}|^2\right) + Ch^2\right].$$

3. Deduce that there exists a constant C such that for sufficiently small h and $t \in [t_n, t_{n+1})$,

$$\mathbb{E}\left(|X_t - X_n|^2\right) \le Ch.$$